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Analyticity of Density of States in a Gauge-Invariant Model for Disordered Electronic Systems

F. Constantinescu,[1] G. Felder,[2] K. Gawedzki,[3] and A. Kupiainen[4]

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The n-orbital gauge-invariant model of disordered electronic systems proposed by Wegner is studied in the regime of dominant diagonal disorder. Analyticity of the density of states is established in two cases: (a) when the number of orbitals is small, (b) when the number of orbitals is large and the energy is in the expected extended states region.

KEY WORDS: Disordered systems; random matrices; Wegner's model.

1. INTRODUCTION

The motion of a particle in a random potential has attracted much attention (for recent reviews see, e.g., Refs. 1 and 2). The most interesting questions for such systems concern the density of states, the nature of the spectrum, and transport properties. A lattice model that has been intensively studied is the Anderson tight binding model with diagonal disorder. The dynamics of the particle is described in this model by the Hamiltonian

$$H = -\Delta + V \qquad \text{on} \quad l^2(\mathbb{Z}^d) \tag{1}$$

where Δ is the finite-difference Laplacian and V is a real, random potential with independent, identically distributed values. In one dimension it was rigorously proved that the density of states has nice regularity properties

[1] Institute for Applied Mathematics Frankfurt am Main University 6000 Frankfurt, Federal Republic of Germany.
[2] Institut des Hautes Etudes Scientifiques, 91440 Bures-sur-Yvette, France.
[3] CNRS, Institut des Hautes Etudes Scientifiques, 91440 Bures-sur-Yvette, France.
[4] Research Institute for Theoretical Physics, Helsinki University, 00170 Helsinki, Finland.

and the spectrum of H is of pure point type with exponentially localized eigenstates.[2] In higher dimensions it was recently shown that the density of states is regular and the spectrum is of pure point type with exponentially localized eigenfunctions for the region of high disorder or large energies.[2] For weak disorder in three dimensions, it has been conjectured by Anderson that there is an interval of (absolutely) continuous spectrum outside which the pure point spectrum appears. The energies at which the type of spectrum changes are called mobility edges. Nevertheless, it is commonly accepted that the density of states does not feel the mobility edges and remains regular all over the spectrum.

At present, the region of continuous spectrum (extended states) of the Anderson model seems to be difficult to control rigorously.

Let us introduce now some notations and give the precise definition of the density of states: Let Λ be a box in $\mathbb{Z}^d$ and $H(\Lambda)$ the Hamiltonian defined in (1) with some boundary conditions for the lattice Laplacian. Let $N_\Lambda(E)$ denote the number of eigenvalues of $H(\Lambda)$ less than or equal to E. The limit

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} N_\Lambda(E) \equiv n(E) \tag{2}$$

exists with probability 1 with respect to the probability distribution $d\lambda(V)$ of the random potential on $\mathbb{Z}^d$ and is independent of V.[2] Furthermore,

$$n(E) = \int \langle x|\, P(E)\, |x\rangle \, d\lambda(V) \tag{3}$$

where x is an arbitrary site of $\mathbb{Z}^d$ and $P(E)$ is the spectral projection onto the subspace of states of energy less than or equal to E. It is known[3] that the integrated density of states $n(E)$ is continuous. If $n(E)$ is differentiable w.r.t. E, then from (3) it follows[1,2] that the density of states is

$$\rho(E) = \frac{d}{dE} n(E) = \frac{1}{\pi} \lim_{\omega \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^d} \operatorname{Im} G_{xx}(\Lambda; E + i\omega) \tag{4}$$

where

$$G_{xx}(\Lambda, E) = \int \langle x|\, (H - E)^{-1}\, |x\rangle \, d\lambda(V) \tag{5}$$

In the present paper we study another model, which has been proposed by Wegner.[4] It is the n-orbital lattice model with nondiagonal disorder and local gauge invariance (for the precise definitions, see

Section 2). Much work has been done on this model by Wegner and co-workers (for review see Ref. 5), including an expansion of the critical exponents at the mobility edge around the lower critical dimensionality two. Schäfer and Wegner[6] have given a Lagrangian formulation of the model by using the replica trick. Instead of the replica trick, one can use the method of superfields worked out for the motion of electrons in random potentials by Efetov.[7] The superfield formulation of the gauge-invariant model will be discussed in Section 2. Models with nondiagonal disorder have been considered rigorously in Ref. 8 (and references given there). It appears that in the cases considered in Ref. 8 the results on the localization problem, known from the Anderson model, remain valid. The gauge-invariant model (with diagonal and off-diagonal Gaussian distribution) considered here is different from that of Ref. 8. It is distinguished by strong phase cancellations induced by the gauge invariance. In the present paper we study analyticity properties of the density of states of Wegner's model for dominant diagonal disorder if (a) the number of orbitals associated to each lattice site is small, (b) the number of orbitals is large and the energy is in the conjectured extended states region.

A particular case of (a) was previously considered by Ziegler.[9]

2. THE *n*-ORBITAL GAUGE-INVARIANT MODEL AND THE AVERAGED ONE-PARTICLE GREEN'S FUNCTION IN THE SUPERFIELD VARIABLES

In the Anderson model described in the introduction the matrix elements G_{xy} of the averaged one-particle Green's function decay rapidly with increasing distance $|x-y|$ due to phase fluctuations. This result was rigorously proved in the region of convergence of the random paths expansion (i.e., large energy or high disorder) in Ref. 10. Wegner introduced[4] a model for the study of disordered electronic systems in which the phases are totally uncorrelated from site to site as a consequence of (local) gauge invariance. Let us give a precise definition of this model. Consider the Hamiltonian $H=V$ on $l^2(\mathbb{Z}^d)\otimes\mathbb{C}^n$ (no deterministic part!) given by the Hermitean matrix elements $V_{xy}^{\alpha\beta}=(V_{yx}^{\beta\alpha})^*$; $x, y\in\mathbb{Z}^d$; $\alpha, \beta=1, 2,\dots, n$; randomly distributed with the Gaussian measure

$$\frac{1}{\text{norm}}\exp\left(-\frac{n}{2}\sum_{x,y}\sum_{\alpha,\beta}\frac{1}{M_{xy}}|V_{xy}^{\alpha\beta}|^2\right)\prod_{\alpha,x}dV_{xx}^{\alpha\alpha}$$
$$\times\prod_{(\alpha,x)\prec(\beta,y)}d\,\mathrm{Re}\,V_{xy}^{\alpha\beta}\,d\,\mathrm{Im}\,V_{xy}^{\alpha\beta}\tag{6}$$

Here $M_{xy}\geqslant 0$ are matrix elements of a symmetric matrix M subjected to some other conditions given below [$\prec$ is an order relation among pairs (α, x)]. We assume that M is a positive-definite, symmetric, translation-invariant matrix with nonnegative elements and $\sum_y M_{xy} \equiv \frac{1}{4}E_0^2 < \infty$, $E_0 > 0$. This implies in particular boundedness of M.

From the requirement of translation invariance we have $M_{xy} = M_{x-y,0} \equiv M_{x-y}$. Assuming that the inverse $w = M^{-1}$ exists, it is positive definite and

$$\sum_{y \in \mathbb{Z}^d} w_{xy} = 4/E_0^2 \tag{7}$$

A typical example for M considered in Section 4 is

$$M = (-\Delta + m^2)^{-1}$$

where Δ is the lattice Laplacian and $m^2 > 0$. In this case $w = -\Delta + m^2$ has finite range. If we restrict the distribution (6) to a single lattice point, we get a (zero-dimensional), n-orbital model which happens to coincide exactly with the Wigner statistical model describing properties of highly excited nuclear levels in compound nucleus.[11] The model (6) can be considered as a hybridization of the Wigner model with a gauge-invariant model obtained from (6) by taking $n = 1$. The local gauge invariance follows from the invariance of (6) under multiplication by phase factors, which can be chosen independently from site to site. The model is called the phase-invariant n-orbital ensemble (PIE).[4] Certainly, the case with V real and symmetric ($V_{xy}^{\alpha\beta} = V_{xy}^{\beta\alpha}$) can also be considered. It is invariant under local reflections and is called the real matrix (n-orbital) ensemble (RME). We shall concentrate here on the PIE. The RME can be studied similarly.

As remarked above, because of phase fluctuations

$$\overline{(V-E)_{xy}^{-1\alpha\beta}} = \overline{(V-E)_{xx}^{-1\alpha\alpha}}\, \delta^{\alpha\beta} \delta_{xy} \tag{8}$$

a relation that will also follow below from an explicit calculation.

In (8) the bar stands for the average with respect to (6). We denote

$$\overline{(V-E)_{xx}^{-1\alpha\alpha}} = G_{xx}(E) = G_{00}(E) \tag{9}$$

The nice feature of the n-orbital model is that, as in the case of the Wigner model, the averaged Green's function $G_{00}(E)$ can be computed exactly in the $n \to \infty$ limit,[4] yielding the well-known semicircle law for the density of states $\rho(E)$ (E real):

$$\rho(E)|_{n=\infty} = \pi^{-1} \operatorname{Im} G_{00}(E+i0) = \begin{cases} 2(E_0^2 - E^2)^{1/2}/\pi E_0^2, & E^2 \leqslant E_0^2 \\ 0, & E^2 \geqslant E_0^2 \end{cases} \tag{10}$$

so that $\pm E_0$ can be identified as the band edges.

We remark that in a certain sense the $n \to \infty$ limit of the n-orbital model resembles the $d \to \infty$ limit of the d-dimensional Anderson model.[4] For $n < \infty$ the density of states no longer vanishes for $E^2 > E_0^2$ and is believed to develop exponential tails. A deep study of the model was undertaken by Wegner and co-workers (for review see Ref. 5). It is believed that for $d > 2$ the model has a central energy interval of extended states separated by mobility edges from regions of localized states as in the Anderson model.

Since analyticity properties of the density of states are the subject of this paper, we remark that it has been rigorously proved[12] that for a large class of models with diagonal or nondiagonal disorder the (averaged) density of states (if it exists) neither vanishes nor diverges inside the band. An exception (which makes the regularity problem for the density of states for nondiagonal disorder rather interesting) seems to be provided by a particular case of RME [two-sublattice models with $d = 1$, $n = 1$, with divergence of $\rho(E)$ at $E = 0$[13]].

Let us sketch the approach to the average one-particle Green's function of PIE based on the superfield formalism and "composite" variables. This material is not new and can be found scattered in the literature of the subject. It is intended as a reader's guide to Sections 3 and 4.

The matrix elements of the Green's function can be expressed with the help of integration over commutative and noncommutative fields φ, φ^* and ψ, ψ^*:

$$(V-E)^{-1\alpha_0\beta_0}_{x_0 y_0} = s \int \varphi^{\alpha_0}_{x_0} \varphi^{*\beta_0}_{y_0} \exp\left\{ -s \sum_{\substack{\alpha,\beta \\ x,y}} \varphi^*_x (V^{\alpha\beta}_{xy} - E\delta^{\alpha\beta}\delta_{xy}) \varphi^\beta_y + \psi^{*\alpha}_x (V^{\alpha\beta}_{xy} - E\delta^{\alpha\beta}\delta_{xy}) \psi^\beta_y \right\} \prod_{\alpha,x} D\varphi^{*\alpha}_x D\varphi^\alpha_x D\psi^{*\alpha}_x D\psi^\alpha_x \tag{11}$$

for $\operatorname{Im} E \neq 0$ and $s = i \operatorname{sgn} \operatorname{Im} E$, where $D\varphi^*_x D\varphi_x$ stays for $(1/\pi)\, d \operatorname{Re} \varphi_x \, d \operatorname{Im} \varphi_x$ and for definiteness the sites x, y are restricted to a finite box $\Lambda \subset \mathbb{Z}^d$.

Computing the V-average, we get

$$\overline{(V-E)^{-1\alpha_0\beta_0}_{x_0\,y_0}}$$

$$= s\int \varphi^{\alpha_0}_{x_0}\varphi^{*\beta_0}_{y_0}\exp\Bigg[sE\sum_{\alpha,x}(\varphi^{*\alpha}_x\varphi^\alpha_x+\psi^{*\alpha}_x\psi^\alpha_x)$$

$$-\frac{1}{2n}\sum_{\substack{\alpha,\beta\\x,y}}(\varphi^{*\alpha}_x\varphi^\beta_y+\psi^{*\alpha}_x\psi^\beta_y)\,M_{xy}(\varphi^{*\beta}_y\varphi^\alpha_x+\psi^{*\beta}_y\psi^\alpha_x)\Bigg]D\varphi^*\,D\varphi\,D\psi^*\,D\psi \tag{12}$$

where

$$D\varphi^*\,D\varphi\,D\psi^*\,D\psi=\prod_{a,x}D\varphi^{*\alpha}_x\,D\varphi^\alpha_x\,D\psi^{*\alpha}_x\,D\psi^\alpha_x$$

Introduce the supermatrix

$$\phi(x)=\sum_\alpha\begin{pmatrix}\varphi^\alpha_x\varphi^{*\alpha}_x & \varphi^\alpha_x\psi^{*\alpha}_x\\ \psi^\alpha_x\varphi^{*\alpha}_x & \psi^\alpha_x\psi^{*\alpha}_x\end{pmatrix} \tag{13}$$

We need below the supertrace of the product $\phi(x)\,\phi(y)$, which is, by definition,[7]

$$\operatorname{Str}\phi(x)\,\phi(y)$$

$$=\sum_{\alpha,\beta}(\varphi^\alpha_x\varphi^{*\alpha}_x\varphi^\beta_y\varphi^{*\beta}_y+\varphi^\alpha_x\psi^{*\alpha}_x\psi^\beta_y\varphi^{*\beta}_y$$

$$-\psi^\alpha_x\varphi^{*\alpha}_x\varphi^\beta_y\psi^{*\beta}_y-\psi^\alpha_x\psi^{*\alpha}_x\psi^\beta_y\psi^{*\beta}_y)$$

$$=\sum_{\alpha,\beta}(\varphi^{*\alpha}_x\varphi^\beta_y+\psi^{*\alpha}_x\psi^\beta_y)(\varphi^{*\beta}_y\varphi^\alpha_x+\psi^{*\beta}_y\psi^\alpha_x) \tag{14}$$

Let now α, γ be real, commuting variables and β, β^* anticommuting variables. Denote by $Q(x)$ the supermatrix

$$Q(x)\equiv(Q^{ij}(x))=\begin{pmatrix}\alpha(x) & \beta^*(x)\\ \beta(x) & i\gamma(x)\end{pmatrix},\qquad i,j=1,2 \tag{15}$$

Note that with $w=M^{-1}$ and using (14), we get

$$\int\exp\Bigg[-\frac{n}{2}\sum_{x,y}\operatorname{Str}Q(x)\,w_{xy}Q(y)$$

$$-s\sum_x\operatorname{Str}Q(x)\,\phi(x)]\,D\alpha\,D\gamma\,D\beta^*\,D\beta$$

$$= \int \exp\left\{-\frac{n}{2}\sum_{x,y} w_{xy}[\alpha(x)\,\alpha(y)+\beta^*(x)\,\beta(y) + \beta^*(y)\,\beta(x)+\gamma(x)\,\gamma(y)] - s\sum_{\alpha,x}[\alpha(x)\,\varphi_x^\alpha\varphi_x^{*\alpha}+\beta^*(x)\,\psi_x^\alpha\varphi_x^{*\alpha}-\beta(x)\,\varphi_x^\alpha\psi_x^{*\alpha}-i\gamma(x)\,\psi_x^\alpha\psi_x^{*\alpha}]\right\} \times D\alpha\, D\gamma\, D\beta^*\, D\beta$$

$$= \exp\left[-\frac{1}{2n}\sum_{\substack{\alpha,\beta\\x,y}} M_{xy}(\varphi_x^\alpha\varphi_x^{*\alpha}\varphi_y^\beta\varphi_y^{*\beta}+2\varphi_x^\alpha\psi_x^{*\alpha}\psi_y^\beta\varphi_y^{*\beta} - \psi_x^\alpha\psi_x^{*\alpha}\psi_y^\beta\psi_y^{*\beta})\right]$$

$$= \exp\left[-\frac{1}{2n}\sum_{x,y} M_{xy}\,\mathrm{Str}\,\phi(x)\,\phi(y)\right] \tag{16}$$

Inserting (16) into (12) produces ($DQ = D\alpha\, D\gamma\, D\beta^*\, D\beta$)

$$\overline{(V-E)^{-1\alpha_0\beta_0}_{x_0y_0}} = s\int \varphi_{x_0}^{\alpha_0}\varphi_{y_0}^{*\beta_0}\exp\left\{-\frac{n}{2}\sum_{x,y} w_{xy}\,\mathrm{Str}\,Q(x)\,Q(y) - s\sum_x \mathrm{Str}[Q(x)-E]\,\phi(x)\right\} DQ\, D\varphi^*\, D\varphi\, D\psi^*\, D\psi \tag{17}$$

We integrate ψ^*, ψ; φ^*, φ first (the change of order in the finite volume Λ can be justified by a simple argument). The ψ^*, ψ integration gives

$$\overline{(V-E)^{-1\alpha_0\beta_0}_{x_0y_0}} = s\int \varphi_{x_0}^{\alpha_0}\varphi_{y_0}^{*\beta_0}\exp\left[-\frac{n}{2}\sum_{x,y} w_{xy}\alpha(x)\,\alpha(y) - n\sum_{x,y} w_{xy}\beta^*(x)\,\beta(y) - \frac{n}{2}\sum_{x,y} w_{xy}\gamma(x)\,\gamma(y)\right] \times \exp\left\{s\sum_{\alpha,x}\frac{\beta(x)\,\varphi_x^{*\alpha}\varphi_x^\alpha\beta^*(x)}{E-i\gamma(x)} + s\sum_{a,x}[E-\alpha(x)]\,\varphi_x^{*\alpha}\varphi_x^\alpha\right\} \times \prod_x \{-s[E-i\gamma(x)]\}^n\, D\varphi^*\, D\varphi\, DQ \tag{18}$$

and the φ^*, φ integration

$$\overline{(V-E)^{-1\alpha_0\beta_0}_{x_0y_0}}$$
$$= -\delta^{\alpha_0\beta_0}\delta_{x_0y_0}\int\left[\sum_u w_{x_0u}\alpha(u)\right]$$
$$\times\exp\left\{-\frac{n}{2}\sum_{x,y} w_{xy}[\alpha(x)\,\alpha(y)+2\beta^*(x)\,\beta(y)+\gamma(x)\,\gamma(y)]\right\}$$
$$\times\prod_x\left[E-\alpha(x)+\frac{\beta(x)\,\beta^*(x)}{E-i\gamma(x)}\right]^{-n}[E-i\gamma(x)]^n\,DQ \qquad (19)$$

To obtain (19), we noticed that, as expected, the Green's function is diagonal and we integrated (18) without the factor

$$\varphi^{\alpha_0}_{x_0}\varphi^{\beta_0}_{y_0}$$

in the integrand and then differentiated the result w.r.t. $\alpha(x_0)$. In the more compact supersymmetric notation, (19) can be written

$$G_{x_0x_0}(A, E)$$
$$= -\int\left[\sum_u w_{x_0u}\alpha(u)\right]$$
$$\times\exp\left[-\frac{n}{2}\operatorname{Str} QwQ - n\operatorname{STr}\ln(E-Q)\right]DQ \qquad (20)$$

where STr acts as Str on the supersymmetric variables and as tr on the lattice sites and as usual Sdet is the superdeterminant[7]

$$\exp[-n\operatorname{Str}\ln(E-Q)]$$
$$=\operatorname{Sdet}(E-Q)^{-n}$$
$$=\prod_x\left[\frac{E-\alpha(x)}{E-i\gamma(x)}+\frac{\beta(x)\,\beta^*(x)}{[E-i\gamma(x)]^2}\right]^{-n}$$
$$=\prod_x\left[\frac{E-i\gamma(x)}{E-\alpha(x)}+\frac{\beta^*(x)\,\beta(x)}{[E-\alpha(x)]^2}\right]^{n}$$
$$=\prod_x\left[\frac{E-i\gamma(x)}{E-\alpha(x)}\right]^n\left\{1+\frac{\beta^*(x)\,\beta(x)}{[E-\alpha(x)][E-i\gamma(x)]}\right\}^n \qquad (21)$$

An equivalent formula is obtained by integrating in (17) directly. As an example, for the particular case $n=1$, we obtain

$$G_{x_0x_0}(\Lambda, E) = -\int \exp\left[-\frac{1}{2}\operatorname{Str} QwQ - \operatorname{STr}\ln(E-Q)\right] \times [(E-Q)^{-1}]^{11}_{x_0x_0}\, DQ \tag{22}$$

where

$$[(E-Q)^{-1}]^{11}_{x_0x_0} = \frac{1}{E-\alpha(x_0)} + \frac{\beta^*(x_0)\,\beta(x_0)}{[E-\alpha(x_0)]^2\,[E-i\gamma(x_0)]} \tag{23}$$

Certainly the anticommuting variables in (20) can be integrated out. The simplest way of doing this is to write down the last factor in the formula (21) as the exponential

$$\exp \sum_x n[E-\alpha(x)]^{-1}\,[E-i\gamma(x)]^{-1}\,\beta^*(x)\,\beta(x)$$

The Gaussian β-integration gives

$$\begin{aligned} G_{x_0x_0}(\Lambda, E) &= -\int \left[\sum_u w_{x_0u}\,\alpha(u)\right] \\ &\quad \times \exp\left\{-\frac{n}{2}\sum_{x,y} w_{xy}[\alpha(x)\,\alpha(y)+\gamma(x)\,\gamma(y)]\right\} \\ &\quad \times \prod_x \left[\frac{i\gamma(x)-E}{\alpha(x)-E}\right]^n \det[n(w-D)]\, D\alpha\, D\gamma \end{aligned} \tag{24}$$

where

$$D_{xy} = \frac{\delta_{xy}}{[\alpha(x)-E][i\gamma(x)-E]}$$

Formally, for $\Lambda \nearrow \mathbb{Z}^d$, using translation invariance and $\sum_y w_{xy} = 4/E_0^2$, one obtains

$$\begin{aligned} G_{00}(E) = -\frac{4}{E_0^2}\int \alpha(0)\exp&\left\{-\frac{n}{2}\sum_{x,y} w_{xy}[\alpha(x)\,\alpha(y)+\gamma(x)\,\gamma(y)]\right. \\ &\left. -n\sum_x \ln\frac{\alpha(x)-E}{i\gamma(x)-E}\right\} \det[n(w-D)]\, D\alpha\, D\gamma \end{aligned} \tag{25}$$

We turn now to the limit of an infinite number of orbitals. For $n \to \infty$ one expects that a saddle-point method will provide good approximation for G_{00}. The saddle-point equations in the (formal) infinite-volume limit are

$$\frac{4}{E_0^2}\alpha + (\alpha - E)^{-1} = \frac{4}{E_0^2} i\gamma + (i\gamma - E)^{-1} = 0 \tag{26}$$

with solutions (independent of x)

$$\alpha = i\gamma = \tfrac{1}{2}E \pm \tfrac{1}{2}(E^2 - E_0^2)^{1/2} \tag{27}$$

We have obtained two saddle points. The integrand has a singularity in the α plane in the upper (lower) half-plane depending on $\operatorname{Im} E$ being positive (negative). The choice of the α-saddle point is dictated by the requirement that the deformation of the α-integration path bringing the saddle point onto it does not cross this singularity. This requirement determines uniquely the saddle point for $|E| < E_0$ as

$$\alpha_c = \tfrac{1}{2}E \mp \tfrac{1}{2}i(E_0^2 - E^2)^{1/2} \qquad \text{for} \quad \operatorname{Im} E = \pm 0$$

Notice integrability in the representation formulas (19), (20), (22), and (25) in fact over any contour $\operatorname{Im}\alpha = \text{const}$, $\operatorname{Im} i\gamma = \text{const}$, $\operatorname{Im}\alpha \operatorname{Im} E < 0$. Translating α and $i\gamma$ by α_c in (19), we obtain

$$\begin{aligned}
&\overline{(V-E)^{-1\alpha_0\beta_0}_{x_0 y_0}} \\
&\quad = -\delta^{\alpha_0\beta_0}\delta_{x_0 y_0} \int \left\{ \sum_u w_{x_0 u}[\alpha(u) + \alpha_c] \right\} \\
&\qquad \times \exp\left\{ -\frac{n}{2} \sum_{x,y} w_{xy}[\alpha(x)\,\alpha(y) + 2\beta^*(x)\,\beta(y) + \gamma(x)\,\gamma(y)] \right\} \\
&\qquad \times \prod_x \left\{ \left(1 + \frac{i\gamma(x)}{\alpha_c - E}\right)^2 \exp\left[-\frac{i\gamma(x)}{\alpha_c - E} + \frac{\alpha(x)}{\alpha_c - E} \right] \right. \\
&\qquad \times \left. \left[\left(1 + \frac{\alpha(x)}{\alpha_c - E}\right)\left(1 + \frac{i\gamma(x)}{\alpha_c - E}\right) + \frac{\beta(x)\,\beta^*(x)}{(\alpha_c - E)^2} \right]^{-1} \right\}^n DQ
\end{aligned} \tag{28}$$

In (28) we integrate separately $\alpha(u)$ and α_c in the sum $\sum_u w_{x_0 u}[\alpha(u) + \alpha_c]$. The constant α_c gives a contribution to the averaged Green's function equal to, for $E^2 \leqslant E_0^2$,

$$G_{00}(E)\,|_{n=\infty} = -\alpha_c \frac{4}{E_0^2} = -\frac{2E}{E_0^2} \pm \frac{2i}{E_0}\left[1 - \left(\frac{E}{E_0}\right)^2\right]^{1/2}, \qquad \operatorname{Im} E = \pm 0 \tag{29}$$

which is exactly the $n = \infty$ Green's function of the model, producing the semicircle density of states (10). The term containing the factor $\alpha(u)$ can be expanded in an asymptotic series in powers of $1/n$. But for achieving a rigorous control of the saddle-point approximation, we have to study the infinite-volume limit of the integral (28).

Formulas (22), (24), and (25) proved in this section will be used in Sections 3 and 4 for the analysis of the corresponding infinite-volume problems by the technique of cluster (high-temperature) expansions. We shall pay special attention to analyticity properties of the density of states.

3. DOMINANT DIAGONAL DISORDER AND ONE ORBITAL PER SITE

Here we assume that $n = 1$ and M has the form

$$M = g(1 + T)^{-1}, \qquad g > 0 \tag{30}$$

where T is a symmetric, translation-invariant matrix satisfying

$$|T_{xy}| \leqslant \varepsilon e^{-\alpha|x-y|} \tag{31}$$

for some positive constants ε, α with ε small. We can suppose that $T_{xx} = 0$ by absorbing the diagonal of T in g. The result is as follows (compare Ref. 9).

Theorem 1. Let $\varepsilon < \varepsilon(\alpha)$. Then for all $g > 0$ and all E the density of states $\rho(E)$ is real analytic in E.

Proof. From (21), (22) we have for $n = 1$

$$G_{xy} \equiv G_{xy}(A, E) = -\int DQ \exp\left(-\frac{1}{2}\operatorname{Str} QwQ\right) \operatorname{Sdet}(E-Q)^{-1} \times [(E-Q)^{-1}]^{11}_{xy} \tag{32}$$

where explicitly

$$\operatorname{Sdet}(E-Q)^{-1} = \prod_x \left[\frac{E - i\gamma(x)}{E-\alpha(x)} + \frac{\beta^*(x)\,\beta(x)}{[E-\alpha(x)]^2}\right] \tag{33}$$

$$[(E-Q)^{-1}]^{11}_{xy} = \delta_{xy}\left(\frac{1}{E-\alpha(x)} + \frac{\beta^*(x)\,\beta(x)}{[E-\alpha(x)]^2\,[E-i\gamma(x)]}\right) \equiv \delta_{xy} F \tag{34}$$

In (32) we expand in T $[w = g^{-1}(1+T)]$ by writing $T_{xy} = T_{xy} s_{xy}|_{s_{xy}=1}$ and applying[14]

$$1 = \prod_{x \neq y} \left(\int_0^1 ds_{xy} \frac{d}{ds_{xy}} + |_{s_{xy}=0} \right) = \sum_{\Gamma} \int ds_\Gamma \, \partial_s^\Gamma |_{s_{\Gamma^c}=0}$$

with $\Gamma \subset \Lambda \times \Lambda$ [and $(x, x) \notin \Gamma$] and $ds_\Gamma = \prod_{(x,y) \in \Gamma} ds_{xy}$, etc. Thus,

$$G_{00} = \sum_{\{X_i\}} \prod_i \gamma(X_i) \tag{35}$$

where $\{X_i\}$ form a partition of Λ and

$$\gamma(X) = \sum_{\Gamma} \int ds_\Gamma \, \partial_\Gamma \int DQ_\Gamma \exp\left(-\frac{1}{2g} \operatorname{Str} Q_\Gamma^2 - \frac{1}{2g} \operatorname{Str} Q_\Gamma T s Q_\Gamma \right) \operatorname{Sdet}(E-Q)^{-1} \cdot \begin{cases} 1, & 0 \notin X \\ F, & 0 \in X \end{cases} \tag{36}$$

with Γ running through connected graphs on X [where $(x, x) \notin \Gamma$ and the line $(x, y) \in \Gamma$ at most with multiplicity one if $x \neq y$] and $Q_\Gamma \equiv Q|_\Gamma$.

Now resum $X \not\ni 0$:

$$G_{00} = \sum_{X \ni 0} \gamma(X) \sum_{\bigcup X_i = \Lambda \backslash X} \prod \gamma(X_i) = \sum_{X \ni 0} \gamma(X) \langle 1 \rangle_{\Lambda \backslash X} = \sum_{X \ni 0} \gamma(X) \tag{37}$$

This is the cluster (high-temperature) expansion for G_{00}.

Note that the resummation produced partition functions which by supersymmetry are equal to one, thus simplifying the cluster expansion. We study now $\gamma(X)$ in two cases:

(i) $X = \{0\}$.

(ii) $|X| > 1$.

In case (i), $\gamma(X)$ reduces to the Green's function of the one-site model,

$$\begin{aligned} \gamma(\{0\}) &= \frac{1}{(2\pi g)^{1/2}} \int_{-\infty}^{\infty} \frac{1}{V-E} \exp\left(-\frac{1}{2g} V^2 \right) dV \\ &= \frac{1}{(2\pi g)^{1/2}} \int_{-\infty}^{\infty} \frac{1}{V-\tilde{E}} \exp\left(-\frac{1}{2} V^2 \right) dV \end{aligned} \tag{38}$$

which can be continued from above or below the real E axis into entire analytic functions (here $\tilde{E} = E g^{-1/2}$).

For $|X| > 1$ use (34) to write

$$\gamma(X) = \sum_{\Gamma} \int ds_{\Gamma}\, \partial_{\Gamma} I(s_{\Gamma}) \tag{39}$$

where (see Section 2 for the details of the computation)

$$\begin{aligned} I(s_{\Gamma}) = -g^{-1/2} \int \prod_{x \in X} \frac{d\alpha(x)\, d\gamma(x)}{2\pi} \exp\left[-\frac{1}{2} (\alpha, (1+Ts)\,\alpha) \right. \\ \left. -\frac{1}{2} (\gamma, (1+Ts)\,\gamma) \right] \det\left[\frac{i\gamma - \tilde{E}}{\alpha - \tilde{E}} (1+Ts) - \frac{\lambda}{(\alpha - \tilde{E})^2} \right] \frac{1}{\alpha(0) - \tilde{E}} \end{aligned} \tag{40}$$

with $\lambda(x) = 1$, $x \neq 0$, $\lambda(x = 0) = 2$. Equation (39) is estimated by the Cauchy formula:

$$I(s_{xy}) = \frac{1}{2\pi i} \int_C \frac{I(t_{xy})}{t_{xy} - s_{xy}}\, dt_{xy}$$

$$C: \quad |t_{xy}| = \frac{1}{A\varepsilon}\, e^{(\alpha/2)|x-y|}$$

$$|\partial_{\Gamma = (x,y)} I(s_{xy})| \leqslant \sup_C |I(t_{xy})| \frac{1}{\inf |t_{xy} - s_{xy}|^2} |t_{xy}|$$

For $\varepsilon < (1/2A)\, e^{\alpha/2}$,

$$\inf |t_{xy} - s_{xy}| > \frac{1}{2\varepsilon A}\, e^{(\alpha/2)|x-y|}$$

so that

$$|\partial_{\Gamma} I(s_{\Gamma})| \leqslant \prod_{(x,y) \in \Gamma} (4A\varepsilon e^{-(\alpha/2)|x-y|}) \sup |I(t_{\Gamma})| \tag{41}$$

with

$$|t_{xy}| = \frac{1}{A\varepsilon}\, e^{(\alpha/2)|x-y|}$$

in the sup. For $I(t_{\Gamma})$, translate the α-contour of integration in (40) to $\alpha \in \mathbb{R} - i$, whence I is analytic in $\operatorname{Im} \tilde{E} > -1/2$ and satisfies there

$$|I(t_{\Gamma})| \leqslant g^{-1/2} O(1)^{|X|} \tag{42}$$

provided we pick $A > A(\alpha)$ such that, say, $\sum_y |T_{xy} t_{xy}| < 1/2$. We used the Hadamard inequality to estimate the determinant in (40). Equations (39)–(42) yield the final estimate

$$|\gamma(X)| \leqslant g^{-1/2} [A(\alpha) \cdot O(1)\, \varepsilon]^{|X|}\, e^{-(\alpha/2)\mathscr{L}(X)} \tag{43}$$

with $\mathscr{L}(X)$ the length of the shortest connected graph on the points of X. Convergence of (35) uniformly in $\varepsilon < \varepsilon(\alpha)$ and A follows and hence the theorem.

We note that Theorem 1 is valid also for $n > 1$ (several orbitals), but in the above estimates, ε will have to tend to zero as $n \to \infty$. We note that we should expect our model to exhibit both extended and localized states in $d > 2$ for M considered above and n small. At least this is plausible for the particular case $n = 1$, $M_{x,y} = 0$ for $|x - y| > 1$, as claimed by Ziegler,[9] who used a heuristic criterion for the existence of an Anderson transition in the gauge-invariant model in three dimensions.[15] A region where one may hope to establish the existence of extended states rigorously for n large will be examined in the next section.

4. DOMINANT DIAGONAL DISORDER AND LARGE NUMBER OF ORBITALS

In this section we study the particular case

$$w = -\Delta + m^2 \tag{44}$$

where Δ is, as usual, the lattice Laplacian and $m > 0$ is large. This is a special case of the model discussed in Section 3, but this time we will derive an analyticity result for the case of large number n of orbitals. As we remember from the discussion in Section 2, the model with $n = \infty$ is exactly solvable and provides us with a semicircular density of states in the energy interval $-E_0 \leqslant E \leqslant E_0$, where $E_0 = 2/m$. We show that for $n < \infty$ but large the density of states will still be analytic in an interval $(-E_0 + \varepsilon, E_0 - \varepsilon)$, where ε is a small number depending on m and n (for $|E|$ large the density of states develops Gaussian tails, as in the Anderson model). In other words, we prove the following result.

Theorem 2. Let $\cos \sigma = E/E_0$, where $E_0 = 2/m$ is the band edge of the $n = \infty$ model and $-E_0 \leqslant E \leqslant E_0$. The density of states is real analytic in E for σ in the interval $(\sigma_0, \pi - \sigma_0)$ for any positive σ_0 provided $m > m_0(\sigma_0)$ and $n > n_0(\sigma_0, m)$.

The proof of Theorem 2 uses saddle-point methods in the thermodynamic limit. In order to understand the particularities of the problem, let us introduce some notations and discuss the general stategy.

We start from formula (24) for the Green's function in finite, say cubic, volume Λ and for $\operatorname{Im} E > 0$. In order to reduce technicalities, we restrict w to Λ by taking $w = -\Delta_\Lambda + m^2$, where Δ_Λ is the lattice Laplacian with the Neumann boundary conditions. Nevertheless, the Dirichlet conditions can also be treated. In (24), we shall translate $\alpha \to \alpha + \alpha_c$, $i\gamma \to i\gamma + \alpha_c$ to the saddle-point regime, where

$$\alpha_c = \frac{E}{2} - \frac{i}{2}(E_0^2 - E^2)^{1/2} \equiv \frac{1}{m} re^{-i\sigma} \tag{45}$$

$r > 0$, σ real. The result is

$$\begin{aligned} G_{00}(\Lambda, E) &= \frac{2}{E_0}\left[-\frac{E}{E_0} + i\left(1 - \frac{E^2}{E_0^2}\right)^{1/2}\right] + G_{00}^{(1)}(V, E) \\ G_{00}^{(1)}(\Lambda, E) &= -\int \left[\sum_{u \in \Lambda} (-\Delta_\Lambda + m^2)_{0u}\, \alpha(u)\right] \\ &\quad \times \exp\left[-\frac{n}{2}(\alpha, -\Delta_\Lambda \alpha) - \frac{n}{2}(\gamma, -\Delta_\Lambda \gamma)\right] \\ &\quad \times \exp\left\{-n \sum_{x \in \Lambda} [U(\alpha(x)) + V(\gamma(x))]\right\} \\ &\quad \times \det\{n[-\Delta_\Lambda + m^2 - F(\alpha, \gamma)]\} \prod_{x \in \Lambda} \frac{d\alpha(x)\, d\gamma(x)}{2\pi} \end{aligned} \tag{46}$$

where

$$U(\alpha) = \tfrac{1}{2} m^2 \alpha^2 + mre^{-i\sigma}\alpha + \ln(1 - mre^{-i\sigma}\alpha) \tag{47}$$

$$V(\gamma) = \tfrac{1}{2} m^2 \gamma^2 - mre^{-i\sigma} i\gamma - \ln(1 - mre^{-i\sigma} i\gamma) \tag{48}$$

$$F(\alpha, \gamma)_{xy} = \left[\alpha(x) - \frac{1}{mr} e^{i\sigma}\right]^{-1} \left[i\gamma(x) - \frac{1}{mr} e^{i\sigma}\right]^{-1} \delta_{xy} \tag{49}$$

In this representation one can take in finite volume the limit $\operatorname{Im} E \searrow 0$, $-E_0 < \operatorname{Re} E < E_0$, which corresponds to $r \nearrow 1$ and $\sigma \in (0, \pi)$. In fact, (46) is analytic in $re^{-i\varphi}$ for $\sigma \neq 0, \pi$, which corresponds to $E \notin (-\infty, -E_0] \cup [E_0, +\infty)$. Our strategy will be to fix $\bar{E} \in (-E_0 + \varepsilon, E_0 - \varepsilon)$ corresponding to $\bar{r} = 1$ and $\bar{\sigma} \in (\sigma_0, \pi - \sigma_0)$ and to set up an expansion of (46) convergent uniformly in the volume and in E, $|E - \bar{E}| \leqslant O(n^{-2})$ with terms analytic in E in this neighborhood. The real analyticity of the density of states claimed in Theorem 2 will then follow immediately.

For $E=\bar{E}$ and large n, the integral in (46) should be dominated by contributions from the saddle points of $U(\alpha)$ and $V(\gamma)$. For $U(\alpha)$, $\alpha=0$ is the only saddle point on the real axis. For $V(\gamma)$, besides $\gamma=0$, there is, however, another real saddle point

$$\gamma_c' = (2/m)\sin\bar{\sigma}$$

Moreover, Re $V(\gamma)=0$ for both $\gamma=0$ and $\gamma=\gamma_c'$. Let us study the stability of $U(\alpha)$ and $V(\gamma)$ around the saddle points. We have

$$\operatorname{Re} U(\alpha) = \tfrac{1}{2}m^2\alpha^2 + m\alpha\cos\bar{\sigma} + \tfrac{1}{2}\ln(1-2m\alpha\cos\bar{\sigma}+m^2\alpha^2) \tag{50}$$

$$\frac{d\operatorname{Re} U(\alpha)}{d\alpha} = \frac{m^2\alpha}{1-2m\alpha\cos\bar{\sigma}+m^2\alpha^2}(m^2\alpha^2 - m\alpha\cos\bar{\sigma} + 2 - 2\cos^2\bar{\sigma}) \tag{51}$$

For $\bar{\sigma}\in[\sigma_1, \pi-\sigma_1]$, where $\sin\sigma_1=\frac{1}{3}$, $\alpha=0$ is the only zero of $d\operatorname{Re} U(\alpha)/d\alpha$ and for some $\eta>0$

$$\operatorname{Re} U(\alpha) \geqslant \eta m^2\alpha^2$$

If we admit now E complex with $|E-\bar{E}|\leqslant O(n^{-2})$, still

$$\operatorname{Re} U(\alpha) \geqslant \eta m^2\alpha^2 - O(n^{-2}) \tag{52}$$

holds, as can be easily seen from (47). For $V(\gamma)$,

$$\operatorname{Re} V(\gamma) = \tfrac{1}{2}m^2\gamma^2 - m\gamma\sin\bar{\sigma} - \tfrac{1}{2}\ln(1-2m\gamma\sin\bar{\sigma}+m^2\gamma^2) \tag{53}$$

and

$$\begin{aligned}\frac{d\operatorname{Re} V(\gamma)}{d\gamma} &= \frac{m^2\gamma}{1-2m\gamma\sin\bar{\sigma}+m^2\gamma^2}(m^2\gamma^2 - 3\gamma\sin\bar{\sigma} + 2\sin^2\bar{\sigma}) \\ &= \frac{m^4\gamma}{1-2m\gamma\sin\bar{\sigma}+m^2\gamma^2}(\gamma-\gamma_c')(\gamma-\gamma_c'')\end{aligned} \tag{54}$$

where

$$\gamma_c' = \frac{2}{m}\sin\bar{\sigma}, \qquad \gamma_c'' = \frac{1}{m}\sin\bar{\sigma}$$

Thus, for $|E-\bar{E}|\leqslant O(n^{-2})$,

$$\operatorname{Re} V(\gamma) \geqslant \eta m^2\gamma^2 + O(n^{-2}) \qquad \text{for} \quad -\infty < \gamma \leqslant \frac{1}{m}\sin\bar{\sigma} \tag{55}$$

$$\operatorname{Re} V(\gamma+\gamma_c')\geqslant \eta m^2\gamma^2+O(n^{-2}) \qquad \text{for} \quad -\frac{1}{m}\sin\bar{\sigma}\leqslant\gamma<\infty \tag{56}$$

Thus, at least for $\bar{\sigma}\in[\sigma_1, \pi-\sigma_1]$, we are faced with the problem of a (symmetric) double-well potential in γ. For the study of the thermodynamic limit this will require a two-phase contour-cluster expansion of the type used in constructive quantum field theory[16,17] (for lattice variants that are useful here see Refs. 18 and 19). The case of $\bar{\sigma}\in(\sigma_0, \sigma_1)\cup(\pi-\sigma_1, \pi-\sigma_0)$ will require a modification of the argument. Since the function $F(\alpha, \gamma)$ of (49) is singular for $\sigma=\pi/2$ (i.e., for real $E=0$), we shall also exclude initially a small interval of $\bar{\sigma}$ around zero, considering only

$$\bar{\sigma}\in[\sigma_1, \pi/2-\sigma_2]\cup[\pi/2+\sigma_2, \pi-\sigma_1] \tag{57}$$

for $\sigma_2>0$ small. Reinclusion of the central interval will be straightforward.

The standard contour-cluster expansion consists first in specifying, for each lattice point, the potential well in which the field lives at this point, and next in performing the cluster expansion, decoupling the nonlocalities of the statistical sum inside islands of each ground state. The minima of $\operatorname{Re} U(\alpha)$ at $\alpha=0$ and of $\operatorname{Re} V(\gamma)$ at $\gamma=0$ and $(2/m)\sin\bar{\sigma}$ are massive (for $\bar{\sigma}$ cutoff from zero and π). Unfortunately, we have to do decouplings also in the determinant in (46) and it is easy to see that whereas $-\Delta+m^2-F(\alpha, \gamma)$ is massive at $\alpha=\gamma=0$, it is *massless* at $\alpha=0$, $\gamma=(2/m)\sin\bar{\sigma}$. This makes decoupling in the right minimum of $\operatorname{Re} V(\gamma)$ impossible. The way out of this difficulty is suggested by the fact that the contribution of the determinant in the right minimum

$$\det[-\Delta+m^2-F(0, (2/m)\sin\bar{\sigma})]=\det(-\Delta) \tag{58}$$

to the integral is smaller than the contribution of the left minimum

$$\det[-\Delta+m^2-F(0,0)]=\det[-\Delta+m^2(1-e^{-2i\sigma})] \tag{59}$$

This is a one-loop effect as compared to the leading order saddle-point contribution and it persists for all $m>0$.

The lesson we learn from this discussion is to set up a countour-cluster expansion with decouplings *only* in the left (fully massive) minimum. The large undecoupled clusters with γ around the right minimum will be outweighed by the phenomenon described above.

In order to prove Theorem 2, it is sufficient to study

$$G_{00}^{(2)}(\Lambda, E)=\int [-m^2\alpha(0)] \exp\left[-\frac{n}{2}(\alpha, -\Delta_\Lambda \alpha)-\frac{n}{2}(\gamma, -\Delta_\Lambda \gamma)\right] \times \exp\left\{-n\sum_{x\in\Lambda}[U(\alpha(x))+V(\gamma(x))]\right\} \times \det\{n[-\Delta_\Lambda+m^2-F(\alpha,\gamma)]\}\prod_{x\in\Lambda}\frac{d\alpha(x)\,d\gamma(x)}{2\pi} \tag{60}$$

with U, V, F given by (47)–(49). Let χ_- be the characteristic function of interval $(-\infty, (1/m)\sin\bar\sigma)$. Then

$$\begin{aligned} 1 &= \prod_{x\in\Lambda}\left[\chi_-(\gamma(x))+\chi_-\left(-\gamma(x)+\frac{2}{m}\sin\bar\sigma\right)\right] \\ &= \sum_{X\subset\Lambda}\prod_{x\in X}\chi_-(\gamma(x))\prod_{x\in\Lambda\backslash X}\chi_-\left(-\gamma(x)+\frac{2}{m}\sin\bar\sigma\right) \end{aligned} \tag{61}$$

Let Σ^* be the set of bonds separating X from $\Lambda\backslash X$ and let Σ be the set of the endpoints of the bonds in Σ^*. We have $\Sigma=\Sigma_-\cup\Sigma_+$, where $\Sigma_-=\Sigma\cap X$, $\Sigma_+=\Sigma\cap(\Lambda\backslash X)$. We insert (61) into (60) and translate $\gamma\to\gamma+h$, where

$$h(x)=\begin{cases}0, & x\in X\\ \gamma_c'=(2/m)\sin\bar\sigma, & x\in\Lambda\backslash X\end{cases} \tag{62}$$

Then

$$G_{00}^{(2)}=\sum_{X\subset\Lambda}G_{00}^{(2)}(X,\Sigma,\Lambda\backslash X) \tag{63}$$

with

$$\begin{aligned} &G_{00}^{(2)}(X,\Sigma,\Lambda\backslash X)\\ &\quad=\int[-m^2\alpha(0)]\prod_{x\in X}\chi_-(\gamma(x))\prod_{x\in\Lambda\backslash X}\chi_-(-\gamma(x))\\ &\quad\times\exp\left[-\frac{n}{2}(h, -\Delta_\Lambda h)-n(\gamma, -\Delta_\Lambda h)-\frac{n}{2}(\alpha, -\Delta_\Lambda\alpha)-\frac{n}{2}(\gamma, -\Delta_\Lambda\gamma)\right]\\ &\quad\times\exp\left\{-n\sum_{x\in\Lambda}[U(\alpha(x))+V(\gamma(x))]\right\}\\ &\quad\times\det\{n[-\Delta_\Lambda+m^2-F(\alpha,\gamma+h)]\}\prod_{x\in\Lambda}\frac{d\alpha(x)\,d\gamma(x)}{2\pi} \end{aligned} \tag{64}$$

We have

$$(h, -\Delta_\Lambda h) = \gamma_c'^2 \, |\Sigma^*|$$

and

$$(\gamma, -\Delta_\Lambda h) = \gamma_c' \left(\sum_{x \in \Sigma_+} b_x \gamma(x) - \sum_{x \in \Sigma_-} b_x \gamma(x) \right)$$

where $1 \leqslant b_x \leqslant 2d$ is the number of bonds in Σ^* ending at x.

In order to decouple (64) inside the left phase region X, we introduce interpolating parameters s_{xy} for the bonds joining points in X. Using, as usual (see Section 3),

$$\begin{aligned} 1 &= \prod_{(x,y) \subset X \setminus \Sigma} \left(\int_0^1 ds_{xy} \frac{\partial}{\partial s_{xy}} + |_{s_{xy}=0} \right) \\ &= \sum_\Gamma \int_0^1 ds_\Gamma \, \partial_s^\Gamma \, |_{s_{\Gamma^c}=0} \end{aligned}$$

(Γ^c is the set of bonds between points of X that are not in Γ), we write

$$\begin{aligned} G_{00}^{(2)} = \sum_{X \subset \Lambda} \int & [-m^2 \alpha(0)] \prod_{x \in X} \chi_-(\gamma(x)) \prod_{x \in \Lambda \setminus X} \chi_-(-\gamma(x)) \\ & \times \exp\left[-\frac{n}{2} \gamma_c'^2 \, |\Sigma^*| - n\gamma_c' \left(\sum_{x \in \Sigma_+} b_x \gamma(x) - \sum_{x \in \Sigma_-} b_x \gamma(x) \right) \right] \\ & \times \sum_{\Gamma \subset X} \int_0^1 ds_\Gamma \, \partial_s^\Gamma \exp\left\{ \left[-\frac{n}{2} (\alpha, -\Delta_\Lambda(s)\, \alpha) - \frac{n}{2} (\gamma, -\Delta(s)\, \gamma) \right] \right. \\ & \left. - n \sum_{x \in \Lambda} [U(\alpha(x)) + V(\gamma(x))] \right\} \\ & \times \det\{ n[-\Delta_\Lambda(s) + m^2 - F(\alpha, \gamma + h)] \} \prod_{x \in \Lambda} \frac{d\alpha(x)\, d\gamma(x)}{2\pi} \end{aligned} \tag{65}$$

where $\Delta_\Lambda(s)$ is the lattice Laplacian with partial decoupling on the bonds of Γ and total decoupling on those of Γ^c:

$$\begin{aligned} (f, -\Delta_\Lambda(s)\, f) = & \sum_{\langle x,y \rangle \in \Gamma} s_{xy} [f(y) - f(x)]^2 \\ & + \sum_{\substack{\langle x,y \rangle \notin \Gamma \cup \Gamma^c \\ x,y \in \Lambda}} [f(y) - f(x)]^2 \end{aligned} \tag{66}$$

Given $X \subset \Lambda$ and Γ, call a bond between points of Λ nondecoupled if it does not belong to Γ^c. Call a subset of Λ connected if all its points can be joined by chains of nondecoupled bonds. Λ falls into a union of connected components Λ_α with all bonds between different components decoupled. Clearly a term in (65) corresponding to given Γ^c factors into a product of contributions from sets Λ_α, which we shall call *polymers*. Moreover, the sums over X and Γ also factor, leading to the expression

$$G_{00}^{(2)} = \sum_{\text{partitions } \{\Lambda_\alpha\}} \prod_\alpha G_{00}^{(3)}(\Lambda_\alpha) \tag{67}$$

where $G_{00}^{(3)}(\Lambda)$ is given by (65) with $\Lambda \to \Lambda_\alpha$ except that (i) the factor $-m^2\alpha(0)$ is dropped if $0 \notin \Lambda_\alpha$, (ii) the sum over $X \subset \Lambda_\alpha$ is constrained by demanding that all points in Λ_α with nearest neighbors in $\Lambda \backslash \Lambda_\alpha$ belong to X, and (iii) we sum over Γ such that Λ_α is connected (i.e., that all its points can be connected by chains of nondecoupled bonds).

The smallest possible polymer is just a one-point set $\{x\}$. Its contribution equals, for $x \neq 0$,

$$\int \chi_-(\gamma(x))\, e^{-n[U(\alpha(x)) + V(\gamma(x))]} \times n[m^2 - F(\alpha(x), \gamma(x))] \frac{d\alpha(x)\, d\gamma(x)}{2\pi} = 1 + O\left(\frac{1}{n}\right) \tag{68}$$

as easily follows from (47)–(49) and stability bounds (52) and (55) via a one-loop calculation. If in (67) we fix the polymer Λ_0 containing zero and resum the others, we do not obtain, as in Section 3, the original partition function in volume $\Lambda \backslash \Lambda_0$ equal to 1. The obstruction is the constraint (ii) above. Hence, we shall proceed somewhat differently. Let us define $G^{(3)}(\Lambda_\alpha)$ in the same way as $G_{00}^{(3)}(\Lambda_\alpha)$ except for dropping factor $-m^2\alpha(0)$ in (65) altogether [thus, $G^{(3)}(\Lambda_\alpha) = G_{00}^{(3)}(\Lambda_\alpha)$ if $0 \notin \Lambda_\alpha$]. Then

$$\sum_{\text{partitions } \{\Lambda_\alpha\}} \prod_\alpha G^{(3)}(\Lambda_\alpha) = 1 \tag{69}$$

as the left-hand side expands the initial partition function (in full volume Λ). Now introduce

$$\tilde{G}_{00}^{(3)}(\Lambda_\alpha) = G_{00}^{(3)}(\Lambda_\alpha) / \prod_{x \in \Lambda_\alpha} G^{(3)}(\{x\}) \tag{70}$$

With this definition, using also (69), we can rewrite (67) as

$$G_{00}^{(2)} = \left[\sum_{\{\Lambda_\alpha\}} \prod_\alpha \tilde{G}_{00}^{(3)}(\Lambda_\alpha) \right] \Big/ \left[\sum_{\{\Lambda_\alpha\}} \prod_\alpha \tilde{G}^{(3)}(\Lambda_\alpha) \right] \tag{71}$$

where now the sums over $\{\Lambda_\alpha\}$ run over families of disjoint subsets $\Lambda_\alpha \subset \Lambda$ containing at least two points (except for $\Lambda_\alpha = 0$, which may appear in the numerator). Equality (71) defines a correlation function of a polymer gas. The control of the thermodynamic limit of (71) is standard,[20,21] provided that

$$|\tilde{G}_{00}^{(3)}(\Lambda_\alpha)| \leqslant \text{const} \times e^{-K|\Lambda_\alpha|} \tag{72}$$

for K big enough.

Inequality (72) is what remains to be shown. It is due to three effects:

1. In X regions the derivatives ∂_{s_Γ} contribute small factors for m large.
2. So do the phase boundaries Σ if n is large, because of the factor $\exp(-\frac{1}{2}n\gamma_c'^2\,|\Sigma^*|)$ in (65).
3. In $\Lambda\backslash X$ small contribution comes from the one-loop effect discussed above; see (58) and (59).

In order to exhibit the last effect, we shall additionally insert in (65) the partition of unity specifying the region $Y \subset \Lambda\backslash X$ in which the fields are in essentially the perturbative regime:

$$\begin{aligned} 1 = \sum_{Y \subset \Lambda\backslash X} \prod_{x \in Y} \chi_1(\alpha(x))\, \chi_1(\gamma(x)) \\ \times \prod_{x \in (\Lambda\backslash X)\backslash Y} [1 - \chi_1(\alpha(x))\, \chi_1(\gamma(x))] \end{aligned}$$

where χ_1 is the characteristic function of the interval $(-n^{-1/3}, n^{1/3})$. We shall also separate the ∂_s^Γ derivatives, according to the Leibnitz rule, into those $\partial_s^{\Gamma_1}$ acting on $\exp[-\frac{1}{2}n(\alpha, -\Delta_{\Lambda_\alpha}(s)\,\alpha)]$, those $\partial_s^{\Gamma_2}$ differentiating $\exp[-\frac{1}{2}n(\gamma, -\Delta_{\Lambda_\alpha}(s)\,\gamma)]$, and those $\partial_s^{\Gamma_3}$ applied to the determinant. The derivatives of $\partial_s^{\Gamma_1}$ and $\partial_s^{\Gamma_2}$ will be explicitly computed, whereas the differentiated determinant will be written with the help of the Cauchy integral formula. All this gives

$$\begin{aligned} G_{00}^{(3)}(\Lambda_\alpha) = \sum_{X, Y, \Gamma_i} \int \prod_{x \in \Lambda_\alpha} \frac{n\, d\alpha(x)\, d\gamma(x)}{2\pi} \\ \times [-m^2\alpha(0)] \prod_{x \in X} \chi_-(\gamma(x)) \prod_{x \in \Lambda_\alpha\backslash X} \chi_-(-\gamma(x)) \\ \times \prod_{x \in Y} \chi_1(\alpha(x))\, \chi_1(\gamma(x)) \prod_{x \in (\Lambda_\alpha\backslash X)\backslash Y} [1 - \chi_1(\alpha(x))\, \chi_1(\gamma(x))] \\ \times \exp\left\{-\frac{n}{2}\gamma_c'^2\,|\Sigma^*| - n\gamma_c'\left[\sum_{x \in \Sigma_+} b_x\gamma(x) - \sum_{x \in \Sigma_-} b_x\gamma(x)\right]\right\} \end{aligned}$$

$$\times\left(-\frac{n}{2}\right)^{|\Gamma_1|+|\Gamma_2|}\prod_{\langle x,y\rangle\in\Gamma_1}[\alpha(y)-\alpha(x)]^2\prod_{\langle x,y\rangle\in\Gamma_2}[\gamma(y)-\gamma(x)]^2$$

$$\times\exp\left\{-\frac{n}{2}(\alpha, -\Delta_{\Lambda_\alpha}(s)\,\alpha)-\frac{n}{2}(\gamma, -\Delta_{\Lambda_\alpha}(s)\,\gamma)\right.$$

$$\left.-n\sum_{x\in\Lambda_\alpha}[U(\alpha(x))+V(\gamma(x)+h(x))]\right\}$$

$$\times\left(\frac{1}{2\pi i}\right)^{|\Gamma_3|}\int\sum_{\langle x,y\rangle\in\Gamma_3}\frac{dt_{xy}}{(t_{xy}-s_{xy})^2}$$

$$\times\det[-\Delta_{\Lambda_\alpha}(s,t)+m^2-F(\alpha,\gamma+h)] \tag{73}$$

where in $-\Delta_{\Lambda_\alpha}(s, t)$ the s_{xy} variables for $\langle x, y\rangle\in\Gamma_3$ were replaced by the integration variables t_{xy} with $|t_{xy}|=m^2/2$.

First, let us estimate the determinant in (73). We shall use a weak version of the Hadamard inequality for (complex) matrices A_{ij},

$$|\det A|\leqslant\prod_i\left(\sum_j|A_{ij}|\right) \tag{74}$$

We need a bound on $m^2-F(\alpha(x),\gamma(x)+h(x))$, where F is given by (49) with $r=1+O(n^{-2})$ and $\sigma=\bar{\sigma}+O(n^{-2})$. A rough estimate gives

$$|m^2-F(\alpha,\gamma(x)+h(x))|\leqslant O(m^2) \tag{75}$$

and consequently

$$\sum_y|(-\Delta_{\Lambda_\alpha}(s,t)(x,y))|+|m^2-F(\alpha(x),\gamma(x)+h(x))|\leqslant O(m^2) \tag{76}$$

However, for $x\in Y$ and thus $|\alpha(x)|, |\gamma(x)|<n^{-1/3}$, we need a better ($m$-independent) estimate:

$$|m^2-F(\alpha(x),\gamma(x)+\gamma_c')|$$

$$=\left|m^2-\left[\alpha(x)-\frac{1}{m}e^{i\sigma}\right]^{-1}\right.$$

$$\left.\times\left[i\gamma(x)-\frac{1}{m}e^{-i\sigma}+O(n^{-2})\right]\right|\leqslant 1$$

if $n^{1/3}\gg m$, say. Since, moreover, for $x\in Y\subset\Lambda_\alpha\backslash X$,

$$-\Delta_{\Lambda_\alpha}(s,t)(x,y)=-\Delta_{\Lambda_\alpha}(x,y)$$

(only the bonds joining points in X are decoupled),

$$\sum_y |(-\Delta_{A_\alpha}(s,t)(x,y)| + |m^2 - F(\alpha(x), \gamma(x)+\gamma_c')| \leqslant 4d+1 \tag{77}$$

if $x \in Y$.

Relations (74)–(77) result in the following bound for the determinant in (73):

$$\left(\frac{1}{2\pi}\right)^{|\Gamma_3|} \left| \int \prod_{\langle x,y\rangle \in \Gamma_3} \frac{dt_{xy}}{(t_{xy}-s_{xy})^2} \det[-\Delta_{A_\alpha}(s,t) + m^2 - F(\alpha, \gamma+h)] \right|$$
$$\leqslant \left(\frac{3}{m^2}\right)^{|\Gamma_3|} O(m^2)^{|A_\alpha \setminus Y|} O(1)^{|Y|} \tag{78}$$

Using this estimate and the stability bounds (52), (55), and (56), we obtain from (73)

$$\begin{aligned}
&|G_{00}^{(3)}(A_\alpha)| \\
&\leqslant \sum_{X,Y,\Gamma_i} \int \prod_x \frac{n\, d\alpha(x)\, d\gamma(x)}{2\pi} [m^2\, |\alpha(0)|] \\
&\times \prod_{x \in (A_\alpha \setminus X)\setminus Y} [1 - \chi_1(\alpha(x))\, \chi_1(\gamma(x))] \\
&\times \exp\left\{ -\frac{n}{2} \gamma_c'^2\, |\Sigma^*| - n\gamma_c' \left[\sum_{x \in \Sigma_+} b_x \gamma(x) - \sum_{x \in \Sigma_-} b_x \gamma(x) \right] \right\} \\
&\times \left(\frac{n}{2}\right)^{|\Gamma_1| + |\Gamma_2|} \prod_{\langle x,y\rangle \in \Gamma_1} [\alpha(y) - \alpha(x)]^2 \\
&\times \prod_{\langle x,y\rangle \in \Gamma_2} [\gamma(y) - \gamma(x)]^2 \\
&\times \exp\left\{ -n\eta m^2 \sum_{x \in A_\alpha} [\alpha(x)^2 + \gamma(x)^2] \right\} \\
&\times \left(\frac{3}{m^2}\right)^{|\Gamma_3|} O(m^2)^{|A_\alpha \setminus Y|} O(1)^{|Y|}
\end{aligned} \tag{79}$$

On the right-hand side, we may replace the factor

$$O(m^2)^{|A_\alpha \setminus Y|} O(1)^{|Y|}$$

by a smaller one,

$$O(m^2)^{|X|} O(1)^{|A_\alpha \setminus X|}$$

diminishing somewhat η in the Gaussian factor, since, due to the characteristic function, for $x \in (\Lambda_\alpha \backslash X) \backslash Y$,

$$\exp\{-n\varepsilon\eta m^2[\alpha(x)^2 + \gamma(x)^2]\} \leqslant \exp(-n^{1/3}\varepsilon\eta m^2) \leqslant O(m^{-2})$$

Now, dropping the last characteristic function, we are left with Gaussian integrals easy to estimate, by separating different terms with the use of the Hölder inequality. Thus, the phase boundaries contribute

$$\exp\left\{-\frac{n}{2}[1 - O(m^{-2})]\,\gamma_0'^2\,|\Sigma^*|\right\} \leqslant \exp\left(-\frac{n}{m^2}\sin^2\bar{\sigma}\,|\Sigma^*|\right)$$

Each s-bond in $\Gamma_1 \cup \Gamma_2$ contributes $O(m^{-2})$, $m^2\,|\alpha(0)|$ contributes (if present) $O(m/n)$, and finally the normalization of the $\alpha(x)$ and $\gamma(x)$ integrals brings a factor $O(m^{-2})$ for each lattice site. Gathering those factors together, we obtain

$$|G_{00}^{(3)}(\Lambda_\alpha)| \leqslant \left[O\left(\frac{m}{n}\right)\right] \sum_{X,Y,\Gamma_i} \left(\exp -\frac{n}{m^2}\sin^2\bar{\sigma}\,|\Sigma^*|\right)$$
$$\times\, O(m^{-2})^{|\Gamma_1| + |\Gamma_2| + |\Gamma_3|}\, O(1)^{|X|}\, O(m^{-2})^{|\Lambda_\alpha \backslash X|}$$

from which (72) follows in a straightforward way for $m > m_0$ and $n > n_0(m)$ if we recall the constraints on X, Y, and Γ_i.

This completes the proof of Theorem 2 for $\bar{\sigma}$ satisfying (57).

Now we extend this result to cover also the central small interval around $E=0$, i.e., $\bar{\sigma} \in (\pi/2 - \sigma_2, \pi/2 + \sigma_2)$, which was excluded above because of the singularity of $F(\alpha, \gamma)$ at $E=0$ and $\gamma = 1/m$. We insert a partition of unity specifying the region Z in which $\gamma(x) \in (1/2m, 3/2m)$, outside of which the previous estimates work. In this region, we combine a single factor

$$\exp[-V(\gamma(x))] = \left\{\exp\left[-\frac{1}{2}m^2\gamma(x)^2 + mre^{-i\sigma}i\gamma(x)\right]\right\}$$
$$\times\,[1 - mre^{-i\sigma}i\gamma(x)]$$

(out of n) for each point with the determinant containing F to see that the F singularity is spurious as

$$|e^{-V(\gamma(x))}[m^2 - F(\alpha, \gamma(x))]| \leqslant O(m^2)$$

for $x \in Z$. The rest of the estimation follows now as before.

In the last step of the proof we shall extend our analysis to the case when $\bar{\sigma} \in (\sigma_0, \sigma_1) \cup (\pi - \sigma_1, \pi - \sigma_0)$ for $\sigma_0 > 0$ arbitrarily small, $m > m_0(\sigma_0)$, and $n > n_0(\sigma_0, m)$. The idea is to rotate the path of α integration, which until now was the real axis, to $e^{-i\varphi}\alpha$, α real, where the angle φ has to be chosen such that the α-stability bound (52) holds for Re $U(e^{-i\varphi}\alpha)$. But [see (47)] (for $E = \bar{E}$)

$$\begin{aligned}\operatorname{Re} U(e^{-i\varphi}\alpha) = \tfrac{1}{2}m^2\alpha^2 \cos 2\varphi + m\alpha \cos(\varphi + \bar{\sigma}) \\ + \tfrac{1}{2}\ln[1 - 2\alpha m \cos(\varphi + \bar{\sigma}) + m^2\alpha^2]\end{aligned}$$

and

$$\begin{aligned}&\frac{d \operatorname{Re} U(e^{-i\varphi}\alpha)}{d\alpha} \\ &\quad = \frac{m^2\alpha}{1 - 2m\alpha \cos(\varphi + \bar{\sigma}) + m^2\alpha^2}[m^2\alpha^2 \cos 2\varphi \\ &\quad + m\alpha \cos(\varphi + \bar{\sigma})(1 - 2\cos 2\varphi) + 1 + \cos 2\varphi - 2\cos^2(\varphi + \bar{\sigma})]\end{aligned}$$

Besides, at $\alpha = 0$, the derivative $d[\operatorname{Re} U(e^{-i\varphi}\alpha)]/d\alpha$ will not change sign if the discriminant

$$\Delta(\varphi, \sigma) = \cos^2(\varphi + \bar{\sigma})(1 + 2\cos 2\varphi)^2 - 4\cos 2\varphi\,(1 + \cos 2\varphi)$$

is nonpositive. This is the case for $\bar{\sigma} \in [\sigma_1, \pi - \sigma_1]$, $\sin \sigma_1 = 1/3$, as we saw before, for the choice $\varphi = 0$. For $\bar{\sigma} \in (0, \sigma_1)$, we may take $\varphi = \pi/6$. Indeed, $\Delta(\pi/6, 0) = 0$ and $\Delta(\pi/6, \bar{\sigma})$ decreases with $\bar{\sigma}$ on $(0, \pi/3)$, since $\cos^2$ is a decreasing function there. An analogous argument holds for $\bar{\sigma} \in (\pi, \pi - \sigma_1)$ with $\varphi = -\pi/6$. Moreover, $e^{-i\varphi} - (1/m)\, e^{i\bar{\sigma}}$ stays cut off from zero during the rotation of α in both cases, assuring the regularity of the integrand. Furthermore, it can be easily checked that for $\bar{\sigma} \in (\sigma_0, \sigma_1) \cup (\pi - \sigma_1, \pi - \sigma_0)$ with σ_0 small and positive, the minimum at $\alpha = 0$ of Re $U(e^{-i\varphi}\alpha)$ stays massive and consequently the stability bound (52) holds after the rotation of α (with η σ_0-dependent). The rest of the analysis proceeds as before, completing the proof of Theorem 2.

5. REMARKS AND CONCLUSIONS

We have proved the analyticity results for the density of states of the gauge-invariant Wegner model for the case of dominant diagonal disorder and either small or large number of orbitals. In particular, the results of Theorem 2 holds for energies $E \in (-E_0 + \varepsilon, E_0 - \varepsilon)$ provided that the number of orbitals is large enough. To our knowledge, this is the first regularity result for the density of states in this region believed to contain extended states in three or more dimensions. The result was obtained by rigorous saddle-point methods applied to the functional integral superfield representation of the averaged resolvent. The semiclassical analysis was combined with a two-phase contour-cluster expansion. The peculiarity of the expansion was that one of the phases had no mass gap on the semiclassical level, but had the energy density shifted up by one-loop effects with respect to the ground-state phase. Consequently, in the cluster expansion, we decoupled the functional measure only in the dominant phase regions. The expansion was carried out for dominant diagonal disorder (large m) (this should not carry us out of the extended states region) and was particularly simple, involving, besides the Hadamard determinant inequality, simple estimates of Gaussian integrals. It is clear that with more technical work the assumption that m is large could be dropped. In particular, the massive phase always outweighs the massless one, as is seen from one-loop computation.

The expected analyticity of the density of states around $E = E_0$ for large but finite n (i.e., also around the mobility edge) seems more difficult. Note that the analyticity fails at $n = \infty$. Another interesting problem would be an extension of the present result to the Anderson model (actually, the case of large dimension might be tractable by similar methods). Finally, to prove rigorously the existence of extended states in the middle of the $n = \infty$ band for three or more dimensions and n large remains the principal open problem for both Wegner's and Anderson's models. Translated into the functional integral language, this requires the control of a massless (Goldstone-type, lattice) field theory.

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